Problem Set 8 (+ Solutions) (Due 3/12/21 11:59PM)

Your Name:

Names of Any Collaborators:

Please review the problem set guidelines in the syllabus before turning in your work. We will discuss the methods for computing the integrals in P3 & P4 on Tuesday 3/9.

P1. This problem gives another formula for the residue at a pole that is sometimes convenient. Suppose that f has a pole of order m at $z_0 \in \mathbb{C}$. Prove that

$$\mathrm{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)].$$

Proof. Since f has a pole of order m at z_0, there exists a function $\varphi(z)$ that is analytic and nonzero at z_0 such that

$$f(z) = \frac{\varphi(z)}{(z-z_0)^m}.$$

By the same theorem, the residue is given by

$$\begin{aligned}
\operatorname*{Res}_{z=z_0} f(z) &= \frac{1}{(m-1)!}\varphi^{(m-1)}(z_0) \\
&= \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]\big|_{z=z_0} \\
&= \frac{1}{(m-1)!}\lim_{z\to z_0}\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)].
\end{aligned}$$

Only the last equality needs to be justified. But this is true because $\varphi(z) = (z-z_0)^m f(z)$ is analytic at z_0 and hence all of it's derivatives are analytic at z_0. In particular, $\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]$ is analytic at z_0, and hence continuous at z_0. Thus,

$$\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]\big|_{z=z_0} = \frac{1}{(m-1)!}\lim_{z\to z_0}\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]$$

by the definition of continuity. This completes the proof. □

P2. For each function $f(z)$, classify all of its isolated singularities as either: removable, essential, or a pole. If an isolated singularity is a pole, make sure so specify the order. Compute the residue at every isolated singularity.[1]

(a) $f(z) = \frac{1}{\sin z}$;

(b) $f(z) = \frac{1}{\sin^2 z}$;

(c) $f(z) = \sin\left(\frac{1}{z}\right)$;

[1] Note: computing a Laurent series is not always the easiest way to identify the type of singularity.

(d) $f(z) = \frac{z}{\cos z}$;

(e) $f(z) = \frac{1}{e^z - 1}$;

(f) $f(z) = ze^{\frac{1}{z-2}}$;

(g) $f(z) = \frac{e^z - 1}{z}$.

Solution. In the following, notice that whenever we use a theorem that relates zeros to pole we DO NOT need to verify that the singularities in question are isolated. This is a consequence of those theorems. However, if we use a Laurent series, we need to show first that the singularity is isolated.

(a) The only singularities occur when $\sin z = 0$, i.e., at the points $z_k = k\pi, k \in \mathbb{Z}$. Writing $p(z) = 1$ and $q(z) = 1$, we see that both p and q are analytic at each z_k, $p(z_k) \neq 0$, $q(z_k) = 0$, and $q'(z_k) = \cos(z_k) = (-1)^k \neq 1$. By a theorem from class, this proves that z_k is a simple pole for all $k \in \mathbb{Z}$ and the residue is given by

$$\operatorname*{Res}_{z=z_k} \frac{1}{\sin z} = \frac{p(z_k)}{q'(z_k)} = (-1)^k.$$

(b) The only singularities occur, as in in (a), at the points $z_k = k\pi, k \in \mathbb{Z}$. Write $p(z) = 1$ and $q(z) = \sin^2 z$. Since $q(z_k) = 0$, $q'(z_k) = 0$ and $q''(z_k) = 2\cos(2z_k) = 1 \neq 0$, we conclude that $q(z)$ has a zero of order $m = 2$ at z_k. Since both p and q are analytic at z_k and also $p(z_k) \neq 0$, we can conclude that $\frac{1}{\sin^2 z} = \frac{p(z)}{q(z)}$ has a pole of order $m = 2$ at z_k. We use P1 to compute the residue. We see that

$$\begin{aligned}
\operatorname*{Res}_{z=z_k} \frac{1}{\sin^2 z} &= \lim_{z \to z_k} \frac{d}{dz} \frac{(z - z_k)^2}{\sin^2 z} \\
&= \lim_{z \to z_k} \frac{2(z - z_k)\sin^2 z - 2(z - z_k)^2 \sin z \cos z}{\sin^4 z} \\
&= 0 && \text{(Use L'Hospital a few times).}
\end{aligned}$$

(c) The only isolated singularity is at $z = 0$. Using the MacLaurin series for sine, we obtain the Laurent series

$$\sin\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)! z^{2n+1}} \quad (0 < |z| < \infty).$$

By definition, then, the singularity is essential and $\operatorname{Res}_{z=0} \sin\left(\frac{1}{z}\right) = 1$.

(d) The singularities occur at $z_k = \pi/2 + k\pi$ for $k \in \mathbb{Z}$. Let $p(z) = z$ and $q(z) = \cos z$. Then both $p(z)$ and $q(z)$ are analytic at each point z_k. Moreover, $p(z_k) \neq 0$, $q(z_k) = 0$, and $q'(z_k) = -\sin z_k = (-1)^k \neq 0$. By the theorem on simple poles, z_k is a simple pole and the residue is given by

$$\operatorname*{Res}_{z=z_k} \frac{z}{\cos z} = -\frac{z_k}{(-1)^k} = (-1)^{k+1} z_k.$$

(e) The singularities occur when $e^z = 1$, that is, at the points $z_k = 2k\pi i$ for $k \in \mathbb{Z}$. Let $p(z) = 1$ and $q(z) = e^z - 1$. Then both $p(z)$ and $q(z)$ are analytic at each point z_k. Moreover, $p(z_k) \neq 0$, $q(z_k) = 0$, and $q'(z_k) = e^{z_k} = 1 \neq 0$. By the theorem on simple poles, each z_k is a simple pole and the residue is given by

$$\operatorname*{Res}_{z=z_k} \frac{1}{e^z - 1} = 1.$$

(f) The only singularity is at $z = 2$. Using the MacLaurin series for the exponential function, we obtain the Laurent series (on $0 < |z-2| < \infty$)

$$\begin{aligned}
ze^{\frac{1}{z-2}} &= 2e^{\frac{1}{z-2}} + (z-2)e^{\frac{1}{z-2}} \\
&= 2\sum_{n=0}^{\infty} \frac{1}{n!(z-2)^n} + (z-2)\sum_{n=0}^{\infty} \frac{1}{n!(z-2)^n} \\
&= 3 + (z-2) + 2\sum_{n=1}^{\infty} \frac{1}{n!(z-2)^n} + \sum_{n=2}^{\infty} \frac{1}{n!(z-2)^{n-1}} \\
&= 3 + (z-2) + 2\sum_{n=1}^{\infty} \frac{1}{n!(z-2)^n} + \sum_{n=1}^{\infty} \frac{1}{(n+1)!(z-2)^n} \\
&= 3 + (z-2) + \sum_{n=1}^{\infty} \frac{2n+3}{(n+1)!(z-2)^n}.
\end{aligned}$$

By definition, $z = 2$ is an essential singularity and the residue is given by

$$\operatorname*{Res}_{z=2} ze^{\frac{1}{z-2}} = \frac{5}{2}.$$

(g) The only singularity is at $z = 0$ and it is isolated since the function is analytic everywhere else. It would be a mistake to try to use a theorem on poles because with $p(z) = e^z - 1$ we have $p(0) = 0$. Instead, we compute the Laurent series about zero. Using the MacLaurin series for the exponential, we obtain

$$\begin{aligned}
\frac{e^z - 1}{z} &= \frac{1}{z}\left(\sum_{n=0}^{\infty} \frac{z^n}{n!} - 1\right) \\
&= \sum_{n=1}^{\infty} \frac{z^{n-1}}{n!} \\
&= \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}.
\end{aligned}$$

Evidently $z = 0$ is a removable singularity and the residue is $\operatorname{Res}_{z=0} \frac{e^z-1}{z} = 0$.

□

P3. Compute the following improper integral using complex analysis:

$$\int_0^{\infty} \frac{x^2}{x^6+1}\, dx.$$

Solution. Let $f(z) = \frac{z^2}{z^6+1}$. The singularities of $f(z)$ are the solutions to $z^6 = -1$. Namely, $z_k = e^{\pi i/6}e^{k\pi i/3}$ for $k = 0, 1, 2, \ldots, 5$. Of course, only z_0, z_1, z_2 have positive imaginary part. Integrating $f(z)$ along the semicircular contour (with $R > 1$) described in the lecture and applying the residue theorem, we obtain

$$\int_{-R}^{R} f(x)\, dx = 2\pi i \sum_{k=0}^{3} \operatorname*{Res}_{z=z_k} f(z) - \int_{C_R} f(z)\, dz.$$

To compute the residues, let $p(z) = z^2$ and $q(z) = z^6 + 1$. Then $p(z_k) \neq 0$, $q(z_k) = 0$ and $q'(z_k) = 6z_k^5 \neq 0$. By the theorem on simple poles, each singularity is a simple pole with residue given by

$$\operatorname*{Res}_{z=z_k} = \frac{z_k^2}{6z_k^5} = -\frac{1}{6}z_k^3.$$

It is easy to see then that

$$2\pi i \sum_{k=0}^{3} \operatorname*{Res}_{z=z_k} f(z) = -\frac{\pi i}{3}(z_0^3 + z_1^3 + z_2^3)$$

$$= -\frac{\pi i}{3}(e^{\pi i/2} + e^{3\pi i/2} + e^{5\pi i/2})$$

$$= -\frac{\pi i}{3}(i + -i + i)$$

$$= \frac{\pi}{3}.$$

Next, we show that $\lim_{R\to\infty} \int_{C_R} f(z)\, dz = 0$. We have

$$\begin{aligned} \left| \int_{C_R} f(z)\, dz \right| &\leq \pi R \max_{|z|=R} \frac{|z|^2}{|z^6 + 1|} \\ &\leq \frac{R^2}{R^6 - 1}. \end{aligned}$$

The right hand side tends to 0 as $R \to \infty$ which proves the claim. Finally, we have

$$\begin{aligned} \int_0^\infty \frac{x^2}{x^6+1}\, dx &= \frac{1}{2}\text{P.V.} \int_{-\infty}^\infty \frac{x^2}{x^6+1}\, dx \qquad \text{(since } f(z) \text{ is even)} \\ &= \frac{1}{2} \lim_{R\to\infty} \int_{-R}^{R} \frac{z^2}{z^6+1}\, dz \\ &= \frac{\pi}{6}. \end{aligned}$$

□

P4. Let $a > 0$. Compute the following improper integral using complex analysis:

$$\int_0^\infty \frac{\cos ax}{x^2+1}\, dx.$$

Solution. Let $f(z) = \frac{1}{z^2+1}$. The singularities of $f(z)e^{aiz}$ are $z = \pm i$. We integrate this function over the semicircula contour (with $R > 1$) described in the lecture. The only singularity that lies interior to the contour is $z = i$. By the residue theorem, we obtain

$$\int_{-R}^{R} f(z)e^{aiz}\, dz = 2\pi i \operatorname*{Res}_{z=i} f(z)e^{aiz} - \int_{C_R} f(z)e^{aiz}\, dz.$$

To compute the residue, let $p(z) = e^{iaz}$ and $q(z) = z^2 + 1$. Then both $p(z)$ and $q(z)$ are analytic at $z = i$. Moreover, $p(i) \neq 0$, $q(i) = 0$ and $q'(i) = 2i \neq 0$. By the theorem on simple poles, the residue is given by

$$\operatorname*{Res}_{z=i} f(z)e^{aiz} = \frac{e^{-a}}{2i}.$$

It is also easily shown that $\lim_{R\to\infty} \int_{C_R} f(z)e^{aiz}\,dz. = 0$. Just apply the triangle inequality for contour integrals as in P3 or Jordan's Lemma with $M_R := \frac{1}{R^2-1}$. Since $\left|\operatorname{Re}\int_{C_R} f(z)e^{aiz}\,dz\right| \leq \left|\int_{C_R} f(z)e^{aiz}\,dz\right|$ we can also conclude that $\lim_{R\to\infty} \operatorname{Re}\int_{C_R} f(z)e^{aiz}\,dz = 0$. Finally, we obtain

$$\begin{aligned}
\int_0^\infty \frac{\cos ax}{x^2+1}\,dx &= \frac{1}{2}\text{P.V.}\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\,dx \\
&= \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R} \frac{\cos ax}{x^2+1}\,dx \\
&= \frac{1}{2}\lim_{R\to\infty}\operatorname{Re}\int_{-R}^{R} f(z)e^{aiz}\,dz \\
&= \frac{1}{2}\lim_{R\to\infty}\left(\operatorname{Re} 2\pi i \operatorname*{Res}_{z=i} f(z)e^{aiz} - \operatorname{Re}\int_{C_R} f(z)e^{aiz}\,dz\right) \\
&= \operatorname{Re}\pi i \frac{e^{-a}}{2i} \\
&= \frac{\pi}{2}e^{-a}.
\end{aligned}$$

□

P5. (**Optional**)[2] Use the argument principle to evaluate the contour integral

$$\int_C \frac{2z^4+3z^2}{2z^5+5z^3+1}\,dz$$

where C is the positively oriented unit circle $|z| = 1$.

Solution. Let $p(z) = 2z^5 + 5z^3 + 1$ and denote by Z the number of zeros, counting orders, of $p(z)$ that lie interior to C. First, we compute Z using Rouchés Theorem. Let $f(z) = 5z^3$ and $g(z) = 2z^5 + 1$. If $|z| = 1$, then

$$|f(z)| = 5|z|^3 = 5$$

and

$$|g(z)| = |2z^5 + 1| \leq 2|z|^5 + 1 = 3.$$

This proves that $|f(z)| = 5 > 3 \geq |g(z)|$ for all $z \in C$. Since also $f(z)$ and $g(z)$ are analytic on and interior to C, we conclude by Rouché that $f(z)$ and $p(z) = f(z) + g(z)$ have the same number of zeros interior to the unit circle. In fact, $f(z) = 5z^3$ has three zeros interior to C. Thus, $Z = 3$.

Now, we compute the integral. Notice that $p'(z) = 10z^4 + 15z^2 = 5(2z^4 + 3z^2)$. Moreover, $p(z)$ is meromorphic interior to C, and analytic and nonzero on C. The fact that $p(z)$ is nonzero on C is a consequence of the fact that $|f(z)| > |g(z)|$ on C (see the proof of Rouché). Hence,

[2] Worth up to 4 points extra credit, added to your score on PSet 8.

the argument principle applies and we have

$$\begin{aligned}
\int_C \frac{2z^4+3z^2}{2z^5+5z^3+1}\,dz &= \frac{1}{5}\int_C \frac{p'(z)}{p(z)}\,dz \\
&= \frac{1}{5}\int_{p(C)} \frac{1}{z}\,dz \\
&= \frac{2\pi i}{5} n(f(C),0) \\
&= \frac{2\pi i}{5} Z && \text{(by the argument principle)} \\
&= \frac{6\pi i}{5}.
\end{aligned}$$

□